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On t-closedness of generalized power series rings

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Abstract

For an extension $A \subseteq B$ of commutative rings, we present a sufficient condition for the ring $[[A^{S,\leqslant}]]$ of generalized power series to be t-closed in $[[B^{S,\leqslant}]]$, where $(S,\leqslant)$ is a torsion-free cancellative ordered monoid. As a corollary, this result can be applied to the ring of power series in any number of indeterminates. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and Preliminaries

Let $A \subseteq B$ be an extension of commutative rings. We say that A is *t-closed* in B if whenever $b^2 - ab, b^3 - ab^2 \in A$ for $a \in A$ and $b \in B$, then $b \in A$. If A is t-closed in B, then A is seminormal in B (i.e., whenever $b^2, b^3 \in A$ for $b \in B$, then $b \in A$). In [6], Onoda et al. introduced t-closedness and investigated the links between t-closedness and quasinormality (an integral domain R is *quasinormal* if $Pic(R) \cong Pic(R[X,X^{-1}])$). They also showed that for an extension $A \subseteq B$ of integral domains such that A is Noetherian and B is finitely generated as an A-module, if A is t-closed in B, then $A[[X]]$ is t-closed in $B[[X]]$. In [2], Benhissi proved that for an extension $A \subseteq B$ of rings such that property $\mathscr{P}_1(A,B)$ holds, if A is t-closed in B, then $A[[X_1,\ldots,X_n]]$ is t-closed in $B[[X_1,\ldots,X_n]]$. For information about the historical development of t-closedness and numerous results about t-closedness, one may consult [7].

In this paper, we will show that the same condition mentioned above also implies that the ring $[[A^{S,\leqslant}]]$ of generalized power series is t-closed in $[[B^{S,\leqslant}]]$, where $(S,\leqslant)$ is a torsion-free cancellative ordered monoid. As an interesting corollary, this result

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can be applied to the ring of power series in any number of indeterminates. Thus, we generalize and unify the results mentioned above.

Let $(S, \leqslant)$ be an ordered set. Recall that $(S, \leqslant)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leqslant)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. It is easy to see that $(S, \leqslant)$ is artinian if and only if every non-empty subset of S has a minimal element. Moreover, if $\leqslant$ is a total order, then $(S, \leqslant)$ is artinian if and only if it is well-ordered. Recall that an ordered monoid $(S, \leqslant)$ is *strictly ordered* if $s, s' \in S$ with $s < s'$, then $s + t < s' + t$ for any $t \in S$. For example, if S is cancellative or the order is trivial, then $(S, \leqslant)$ is a strictly ordered monoid.

Let $(S, \leqslant)$ be a strictly ordered monoid and let D be a commutative ring with 1. Let $R = [[D^{S,\leqslant}]]$ be the set of all functions $f : S \to D$ such that $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. We call $\{f(s) \mid s \in supp(f)\}$ the set of all coefficients of f. It is clear that R is an additive abelian group with pointwise addition. For every $s \in S$ and $f_1, \ldots, f_n \in R$, let $X_s(f_1, \ldots, f_n) = \{(u_1, \ldots, u_n) \in S^n \mid s = u_1 + \cdots + u_n,\ u_i \in supp(f_i)$ for each $i\}$. It follows from [3, (e) p. 368] that $X_s(f_1, \ldots, f_n)$ is finite. This fact allows one to define the operation of convolution $*$ as follows:

$$(f * h)(s) = \sum_{(u,v) \in X_s(f,\, h)} f(u)h(v).$$

With this operation, and pointwise addition, R becomes a commutative ring with identity element e, where

$$e(s) = \begin{cases} 1, & \text{if } s = 0, \\ 0, & \text{if } 0 \neq s \in S. \end{cases}$$

We call R the *ring of generalized power series*. It should be noted that the definition of $[[D^{S,\leqslant}]]$ depends on the order $\leqslant$, for example, see [3, p. 371]. Following [8, (2.5)], R is an integral domain if and only if D is an integral domain and S is torsion-free and cancellative. It follows from [3, p. 368] that D is canonically embedded as a subring of $[[D^{S,\leqslant}]]$, and that S is canonically embedded as a submonoid of $([[D^{S,\leqslant}]] \setminus \{0\}, *)$. Numerous examples of rings of generalized power series are given in [8,9].

In [3,8–11], there are many results on ordered monoids and the rings of generalized power series. The following result is well-known and will be frequently used in the sequel. If S is a torsion-free cancellative monoid and if $\leqslant$ is any compatible strict order on S, then there exists a compatible strict total order $\leqslant'$ on S which is finer than $\leqslant$ (i.e., if $s, t \in S$ such that $s \leqslant t$, then $s \leqslant' t$). General references for any undefined terminology or notation are [3,8–11].

2. Main results

Let $A \subseteq B$ be an extension of commutative rings and $n \geqslant 1$. Recall from [1] that property $\mathscr{P}_n(A, B)$ holds if for each $a \in A$ and $b \in B$ such that $nab \in A$, we have $nab^2 \in A$. In [1], Anderson et al. investigated property $\mathscr{P}_n(A,B)$ because of its relationship to root closedness of power series ring. Among other results, it was shown that a commutative ring A is p-injective (i.e., each principal ideal of A is an annihilator of some subset of A) if and only if property $\mathscr{P}_1(A,B)$ is satisfied for any ring extension B of A ([1, Corollary 1.15]), and that a reduced commutative ring A is von Neumann regular if and only if property $\mathscr{P}_1(A,B)$ is satisfied for any ring extension B of A ([1, Proposition 1.14]).

The following definition (with notation π) is from [11, pp. 571]. If $0 \neq f \in [[D^{S,\leqslant}]]$, we denote by $\mathcal{O}(f)$ the set of minimal elements of *supp*(f). Then $\mathcal{O}(f)$ is a non-empty finite set consisting of pairwise order incomparable elements. If $\mathcal{O}(f)$ consists only of one element s, then we write $\mathcal{O}(f) = s$.

Lemma 2.1. *Let $A \subseteq B$ be commutative rings such that A is seminormal in B, and let $k \geqslant 2$ be an integer. Then the following assertions are equivalent*:

(1) *Property $\mathscr{P}_1(A,B)$ is satisfied.*

(2) *If $a \in A$, $b \in B$, and $ab \in A$, then $ab^i \in A$* for each $i \geqslant 1$.

(3) *If $a \in A$, $b \in B$,* and *$a^k b \in A$, then $ab \in A$.*

(4) *If $a \in A$, then $(A :_B a) := \{b \in B \mid ab \in A\}$ is an A-subalgebra of B.*

(5) *Let $(S, \leqslant)$ be a torsion-free cancellative ordered monoid. If $f \in [[A^{S,\leqslant}]], g \in [[B^{S,\leqslant}]]$ and $fg \in [[A^{S,\leqslant}]]$,* then *$f(s)g(t) \in A$ for all $s,t \in S$*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): [2, Lemma 1]. (3) $\Rightarrow$ (4): It is clear that $(A :_B a)$ is an A-submodule of B. Assume that (3) holds. If $b_1, b_2 \in (A :_B a)$, then $ab_1, ab_2 \in A$, and hence $a^2(b_1b_2) \in A$. Thus, by (3), we have $ab_1b_2 \in A$, and hence $b_1b_2 \in (A :_B a)$. Thus $(A :_B a)$ is an A-subalgebra of B. (4) $\Rightarrow$ (1): Easy. (5) $\Rightarrow$ (1): Assuming (5), let $a \in A, b \in B$ such that $ab \in A$. Considering the power series $f = a(1 - bX) \in A[[X]]$ and $g = 1/1 - bX \in B[[X]]$, we obtain that $ab^2 \in A$. Hence property $\mathscr{P}_1(A,B)$ is satisfied.

(3) $\Rightarrow$ (5): Assume that (3) holds. Since S is torsion-free and cancellative, there exists a compatible strict total order $\leqslant'$ on S which is finer than $\leqslant$. Note that $[[A^{S,\leqslant}]]$ (resp., $[[B^{S,\leqslant}]]$) is a subring of $[[A^{S,\leqslant'}]]$ (resp., $[[B^{S,\leqslant'}]]$). Thus we may assume that the order $\leqslant$ is total.

Step I: Let $\mathcal{O}(f) = s_0$. Then we will show that $f(s_0)g(t) \in A$ for all $t \in S$. If $\mathcal{O}(g) = t_0$, then clearly $f(s_0)g(t_0) = (fg)(s_0 + t_0) \in A$ by hypothesis. Assume that $f(s_0)g(v) \in A$ for all $v \in$ *supp*(g) such that $t_0 \leqslant v < w$. Then,

$$(f(s_0))^2 g(w) = f(s_0)[(fg)(s_0 + w)] - f(s_0) \left(\sum_{\substack{(y_1,y_2) \in X_{s_0+w} \\ y_1 \neq s_0,\ y_2 \neq w}} f(y_1)g(y_2) \right) \in A.$$

By (3) (with $k = 2$), we have $f(s_0)g(w) \in A$, and hence $f(s_0)g(t) \in A$ for all $t \in S$.

Step II: We will show that $f(s)g(t) \in A$ for all $s,t \in S$. Assume that $f(v')g(t) \in A$ for all $v' \in supp(f)$ such that $s_0 \leqslant v' < u$ and all $t \in S$. Define f_u as follows:

$$f_u(x) = \begin{cases} f(x) & \text{if } x < u, \\ 0 & \text{if } u \leqslant x. \end{cases}$$

Then $f - f_u, (f - f_u)g \in [[A^{S,\leqslant}]]$ (by induction hypothesis), and $\mathcal{O}(f - f_u) = u$. Applying Step I, we obtain that $f(u)g(t) \in A$ for all $t \in S$. Thus $f(s)g(t) \in A$ for all $s,t \in S$. $\square$

Lemma 2.2. *Let $A \subseteq B$ be commutative rings satisfying property $\mathscr{P}_1(A,B)$ such that A is seminormal in B. Let $(S, \leqslant)$ be a torsion-free cancellative ordered monoid and let $f,h \in [[B^{S,\leqslant}]]$ such that $fh, f^2h, fh^2 \in [[A^{S,\leqslant}]]$. Let C be the A-subalgebra with identity of B generated by the coefficients of f and of h. Then for each $s,t \in S$ we have $Cf(s)h(t) \in A$.*

Proof. As shown in the proof of Lemma 2.1, we may assume that the order $\leqslant$ is total. Assume that the assertion is false, and let s be the first element in $supp(f)$ such that $Cf(s)h(t) \nsubseteq A$ for some $t \in S$. Assume that t is minimal. Since $fh, f^2h, fh^2 \in [[A^{S,\leqslant}]]$, we have $Cfh \subseteq [[A^{S,\leqslant}]]$ by Lemma 2.1, in particular, $cf(s)[(fh)(s+t)] \in A$ for all $c \in C$. It is now easy to prove that $cf(s)^2h(t) \in A$ for all $c \in C$. Indeed, by the choice of s and the minimality of t, we have that

$$\begin{aligned} &cf(s)^2h(t) \\ &\quad = cf(s)\left((fh)(s+t) - \left(\sum_{\substack{(u,v)\in X_{s+t} \\ u<s}} f(u)h(v)\right) - \left(\sum_{\substack{(u,v)\in X_{s+t} \\ v<t}} f(u)h(v)\right)\right) \in A. \end{aligned}$$

Thus $(cf(s)h(t))^n \in A$ for all $c \in C$ and $n \geqslant 2$. Since A is seminormal in B, we obtain that $Cf(s)h(t) \subseteq A$, a contradiction. $\square$

Theorem 2.3. *Let $(S, \leqslant)$ be a torsion-free cancellative ordered monoid and let $A \subseteq B$ be commutative rings satisfying property $\mathscr{P}_1(A,B)$. If A is t-closed in B, then the generalized power series ring $[[A^{S,\leqslant}]]$ is t-closed in $[[B^{S,\leqslant}]]$.*

Proof. Let $f \in [[B^{S,\leqslant}]]$ and $g \in [[A^{S,\leqslant}]]$ such that $f^2 - gf, f^3 - gf^2 \in [[A^{S,\leqslant}]]$. Let $h := f - g \in [[B^{S,\leqslant}]]$. Then $fh, f^2h, fh^2 (= f^2h - fgh) \in [[A^{S,\leqslant}]]$. Applying Lemma 2.2, we have $f(s)(f(s) - g(s)), f(s)^2(f(s) - g(s)) \in A$, and hence $f(s) \in A$ for all $s \in S$ since A is t-closed in B. Thus $f \in [[A^{S,\leqslant}]]$, and so $[[A^{S,\leqslant}]]$ is t-closed in $[[B^{S,\leqslant}]]$. $\square$

Let $A \subseteq B$ be an extension of commutative rings. Then property $\mathscr{P}_1(A,B)$ is satisfied if either of the following conditions is satisfied:

- B is an integral extension of A and A is seminormal in B [2, Lemma 6].
- A is an integral domain with quotient field K and $K \cap B = A$ [2, Corollary 8].

Let A be a subring of a commutative ring B such that A is an integral domain with quotient field K. Then $K \cap B = A$ if each principal ideal of A is contracted from B (i.e., $aB \cap A = aA$ for each $a \in A$), in particular, if B is a faithfully flat A-module.

Corollary 2.4. *Let $(S, \leqslant)$ be a torsion-free cancellative ordered monoid and let $A \subseteq B$ be an extension of commutative rings. Assume that A is t-closed in B. Then the generalized power series ring $[[A^{S,\leqslant}]]$ is t-closed in $[[B^{S,\leqslant}]]$ if one of the following conditions is satisfied*:

(1) *A is von Neumann regular.*
(2) *A is p-injective.*
(3) *B is an integral extension of A.*
(4) *A is an integral domain with quotient field K and $K \cap B = A$.*

Recall from [3] that an ordered monoid $(S, \leqslant)$ is *subtotally ordered* if for every $s \in S$ there exists an integer $k \geqslant 1$ such that $ks \leqslant 0$ or $0 \leqslant ks$. Clearly a totally ordered monoid is subtotally ordered.

As usual, we denote by R^* the complete integral closure of the commutative ring R. Let S be a torsion-free cancellative monoid with quotient group G. Let $S^* = \{g \in G \mid \text{there exists } s \in S \text{ such that } s + ng \in S \text{ for all } n \geqslant 1\}$; S^* is called the *complete integral closure of S*. We say that S is *completely integrally closed* if $S^* = S$.

Theorem 2.5 (cf. [11, (5.2)]). *Let A be an integral domain. Let $(S, \leqslant)$ be a torsion-free cancellative subtotally ordered monoid. Then $([[A^{S,\leqslant}]])^* \subseteq [[(A^*)^{S^*,\leqslant}]]$. Moreover, if A and S are completely integrally closed, then the ring $[[A^{S,\leqslant}]]$ of generalized power series is completely integrally closed.*

Proof. Let K be the quotient field of A. Let G be the quotient group of S endowed with the natural extension of the order of S. Since $(G, \leqslant)$ is a subtotally ordered torsion-free group, $F := [[K^{G,\leqslant}]]$ is a field by [3], and hence it contains the quotient field L of $[[A^{S,\leqslant}]]$.

Let $f \in ([[A^{S,\leqslant}]])^* \subseteq F$. Then there exists $0 \neq h \in [[A^{S,\leqslant}]]$ such that $h * f^n \in [[A^{S,\leqslant}]]$ for all $n \geqslant 1$. We will show the following:

Claim: $f \in [[(A^*)^{S^*,\leqslant}]]$. *That is,* $v \in S^*$ and $f(v) \in A^*$ *for every* $v \in supp(f)$.

Since G is torsion-free, there exists a compatible strict total order $\leqslant'$ on G which is finer than $\leqslant$. Note that $[[A^{S,\leqslant}]]$ is a subring of $[[A^{S,\leqslant'}]]$. Let $\mathcal{O}'(f) = s$ and $\mathcal{O}'(h) = t$. By [11, (1.17)], for every $n \geqslant 1$, $\mathcal{O}'(h * f^n) = t + ns \in S$ since $h * f^n \in [[A^{S,\leqslant}]]$. Thus $s \in S^*$.

Similarly, $(h * f^n)(t + ns) = h(t)f^n(ns) = h(t)(f(s))^n \in A$ for every $n \geqslant 1$. Thus $f(s) \in A^*$.

Assume that the Claim holds for every $w \in supp(f)$ such that $s \leqslant' w <' u$. Define f_u as in the proof of Lemma 2.1 (3) $\Rightarrow$ (5). Then

$$h*(f-f_u)^n = h*f^n - \binom{n}{1} h*f^{n-1}*f_u + \binom{n}{2} h*f^{n-2}*(f_u)^2 - \cdots$$
$$+(-1)^n h*(f_u)^n \in [[A^{S,\ \leqslant}]]$$

by hypothesis.

Since $\mathcal{O}'(f-f_u)=u$, we have $u \in S^*$ as in the $\mathcal{O}'(f)=s$ case. Similarly, $(h*(f-f_u)^n)(t+nu)=h(t)(f-f_u)^n(nu)=h(t)(f(u))^n \in A$ for every $n \geqslant 1$. Thus $f(u) \in A^*$.

Therefore, $v \in S^*$ and $f(v) \in A^*$ for every $v \in supp(f)$, and so $f \in [[(A^*)^{S^*,\ \leqslant}]]$. The rest of the proof is clear. □

The proof of the following theorem is similar to that of [2, Proposition 9]. For completeness, we will give its proof.

Theorem 2.6. *Let A be an integral domain and let $(S, \leqslant)$ be a torsion-free cancellative subtotally ordered monoid. Then $[[A^{S,\ \leqslant}]]$ is t-closed in $[[(A^*)^{S^*,\ \leqslant}]]$ if and only if $[[A^{S,\ \leqslant}]]$ is t-closed in its quotient field.*

Proof. ($\Leftarrow$): This follows from [2, Lemma 4].

($\Rightarrow$): It follows from Theorem 2.5 that $([[A^{S,\ \leqslant}]])^* \subseteq [[(A^*)^{S^*,\ \leqslant}]]$. That $[[A^{S,\ \leqslant}]]$ is t-closed in its quotient field follows from the two facts that $[[A^{S,\ \leqslant}]]$ is t-closed in $([[A^{S,\ \leqslant}]])^*$ and $([[A^{S,\ \leqslant}]])^*$ contains the integral closure of $[[A^{S,\ \leqslant}]]$. □

Corollary 2.7. *Let A be an integral domain such that the property $\mathscr{P}_1(A, A^*)$ holds. Let $(S, \leqslant)$ be a torsion-free cancellative subtotally ordered monoid such that S is completely integrally closed. Then A* is *t-closed in A^* if and only if $[[A^{S,\ \leqslant}]]$ is t-closed in its quotient field.*

As mentioned in [4], there are at least three distinct rings of power series in infinitely many indeterminates $\{X_\lambda\}_{\lambda \in \Lambda}$ in the literature. That is,

- $D[[\{X_\lambda\}]]_1 := \lim\limits_{\to\, F \subset \Lambda} D[[\{X_\lambda\}_{\lambda \in F}]]$, where F is a finite subset of Λ.
- $D[[\{X_\lambda\}]]_2 :=$ the completion of $D[\{X_\lambda\}_{\lambda \in \Lambda}]$ for the $(\{X_\lambda\}_{\lambda \in \Lambda})$-adic topology.
- $D[[\{X_\lambda\}_{\lambda \in \Lambda}]]_3 :=$ the set of all functions $f : \bigoplus_\Lambda \mathbb{N} \to D$ with pointwise addition and the convolution $*$. This ring is called the *full* ring of power series.

Note from [4, p. 543] that $D[[\{X_\lambda\}]]_1 \subseteq D[[\{X_\lambda\}]]_2 \subseteq D[[\{X_\lambda\}]]_3$ (up to isomorphism), and that each of these containments is proper (if and only if Λ is infinite). In [4, Section 2], it was shown that for any integral domain D, $D[[\{X_\lambda\}]]_3 \cap K_i = D[[\{X_\lambda\}]]_i$, where K_i denotes the quotient field of $D[[\{X_\lambda\}]]_i$.

The following observation is from [4]: Let $S = \bigoplus_\Lambda \mathbb{N}$, where the indexing set Λ has infinite cardinality. For $s = (n_\lambda)_{\lambda \in \Lambda} \in S$, we define $\sigma(s) = \sum_\lambda n_\lambda$. Given a well-ordering on the set Λ, we well-order the set S as follows: If $s = (m_\lambda)$ and $t = (n_\lambda)$ are distinct elements of S, then $s <_\sigma t$ if $\sigma(s) < \sigma(t)$ or if $\sigma(s) = \sigma(t)$ and $m_\lambda < n_\lambda$ for the first $\lambda \in \Lambda$ such that $m_\lambda \neq n_\lambda$. Then the order $<_\sigma$ is compatible with the semigroup operation. Also $(S, <_\sigma)$ is a torsion-free cancellative strictly totally ordered monoid. Let D be an integral domain. Then $D[[\{X_\lambda\}_{\lambda\in\Lambda}]]_3 = [[D^{S,\leqslant_\sigma}]]$.

Corollary 2.8. *Let $A \subseteq B$ be integral domains satisfying property $\mathscr{P}_1(A, B)$. If A is t-closed in B, then $A[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ is t-closed in $B[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ for each $i = 1, 2, 3$.*

Proof. The case $i = 3$ follows from Theorem 2.3. Thus the cases $i = 1$ and $i = 2$ follow easily from the fact that $D[[\{X_\lambda\}]]_3 \cap K_i = D[[\{X_\lambda\}]]_i$, where K_i denotes the quotient field of $D[[\{X_\lambda\}]]_i$. $\square$

Lemma 2.9 (Liu [5, Proposition 2.8]). Let $(S, \leqslant)$ *be a torsion-free cancellative ordered monoid. Assume that $A \subseteq B$ is an extension of commutative rings such that A is seminormal in B, and let $n \geqslant 1$. Then properties $\mathscr{P}_n(A,B)$ and $\mathscr{P}_n([[A^{S,\leqslant}]], [[B^{S,\leqslant}]])$ are equivalent.*

Corollary 2.10. *Let $(S_1,\leqslant_1),\ldots,(S_m,\leqslant_m)$ be torsion-free cancellative ordered monoids. Denote by $(lex \leqslant_i)$* and *$(rev\ lex \leqslant_i)$ the lexicographic order and the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_m$. Let $A \subseteq B$ be commutative rings satisfying property $\mathscr{P}_1(A, B)$. Assume that A is t-closed in B. Then $[[A^{S_1\times\cdots\times S_m,(lex\leqslant_i)}]]$ and $[[A^{S_1\times\cdots\times S_m,(rev\ lex\leqslant_i)}]]$ are t-closed in $[[B^{S_1\times\cdots\times S_m,(lex\leqslant_i)}]]$ and $[[B^{S_1\times\cdots\times S_m,(rev\ lex\leqslant_i)}]]$, respectively.*

For an ideal I of a commutative ring A, let $[[I^{S,\leqslant}]]{:=}\{f \in [[A^{S,\leqslant}]] \mid f(s) \in I$ for every $s\in S\}$; then $[[I^{S,\leqslant}]]$ is an ideal of $[[A^{S,\leqslant}]]$ and $[[A^{S,\leqslant}]]/[[I^{S,\leqslant}]] \cong [[(A/I)^{S,\leqslant}]]$ by [8, (2.2)].

The following result can be easily verified along the lines of the proof of [2, Proposition 13] by using the remark just above and Theorem 2.3.

Theorem 2.11. *Let $A \subseteq B$ be commutative rings with a common ideal I and let $(S, \leqslant)$ be a torsion-free cancellative ordered monoid. Then:*
(1) *$[[A^{S,\leqslant}]]$ is t-closed in $[[B^{S,\leqslant}]]$ if and only if $[[(A/I)^{S,\leqslant}]]$ is t-closed in $[[(B/I)^{S,\leqslant}]]$.*
(2) *If I is a common maximal ideal, then the following assertions are equivalent:*
(i) *A is t-closed in B.*
(ii) *A/I is t-closed in B/I.*
(iii) *$[[A^{S,\leqslant}]]$ is t-closed in $[[B^{S,\leqslant}]]$.*
(iv) *$[[(A/I)^{S,\leqslant}]]$ is* t-closed in $[[(B/I)^{S,\leqslant}]]$.

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